Journal of Global Optimization **21:** 445–455, 2001.

Global Optimality Conditions in Maximizing a Convex Quadratic Function under Convex Quadratic Constraints ⋆

JEAN-BAPTISTE HIRIART-URRUTY
Laboratoire MIP, UMR 5640, U.F.R. Mathématiques, Informatique, Gestion, Université Paul Sabatier, 118 Route de Narbonne, 31062 Topulouse Cedex 4, France (E-mail: jbhu@cict.fr)

Abstract. For the problem of maximizing a convex quadratic function under convex quadratic constraints, we derive conditions characterizing a globally optimal solution. The method consists in exploiting the global optimality conditions, expressed in terms of ε-subdifferentials of convex functions and ε-normal directions, to convex sets. By specializing the problem of maximizing a convex function over a convex set, we find explicit conditions for optimality.

Key words: global optimality condition, quadratic objective function, convex quadratic constraints

1. Introduction

Optimization problems, where all the data (objective function and constraint functions) are quadratic functions, cover a large spectrum of situations; they constitute an important part in the field of optimization, see [1, section 8] for a recent survey on the subject. Tackling them from the (global) optimality and duality viewpoints is not as yet at hand. We consider here a special class of such optimization problems, with a convex objective function and convex inequality constraints:

$$(\mathcal{P})\quad \begin{cases} \text{Minimize (or maximize)} & f(x) := \frac{1}{2}\langle Ax, x\rangle + \langle a, x\rangle + \alpha \\ \text{subject to} & \\ g_i(x) := \frac{1}{2}\langle Q_i x, x\rangle + \langle b_i, x\rangle + c_i \leqslant 0 & \text{for all } i = 1, \ldots, m, \end{cases}$$

where A, $Q_1, \ldots, Q_m$ are positive semidefinite symmetric n-by-n matrices, a, $b_1, \ldots, b_m$ vectors in $\mathbb{R}^n$ and $\alpha, c_1, \ldots, c_m$ real numbers. The notation $\langle ., .\rangle$ stands for the standard inner product in $\mathbb{R}^n$.

In such a setting, two situations are well understood: when there is only *one* inequality constraint, or when the f to be *minimized* as well as the constraint functions g_i are convex (hence $(\mathcal{P})$ is a *convex minimization* problem). When there is only one inequality constraint, surprisingly enough, the usual first-order Karush–Kuhn–Tucker (KKT) conditions can be complemented so as to provide a characterization of global solutions to $(\mathcal{P})$ ([6,8]): under a slight assumption on the

⋆ A preliminary version of this paper was presented at the French–Belgian–German Conference on Optimization, September 1998 constraint function g_1, the KKT conditions at $\bar{x}$ with an associated Lagrange–KKT multiplier $\bar{\mu} \geqslant 0$, complemented with the condition '$A + \bar{\mu} Q_1$ is positive semidefinite', characterize a global solution to $(\mathcal{P})$. In such a case one can assert that, roughly speaking, $(\mathcal{P})$ is a convex problem in a hidden form. When all the matrices $A, Q_1, \ldots, Q_m$ are positive semidefinite and $(\mathcal{P})$ consists in minimizing the quadratic convex f under quadratic convex inequalities $g_i(x) \leqslant 0$, $i = 1, \ldots, m$, $(\mathcal{P})$ is a particular convex minimization problem for which one knows the optimality conditions.

In the present work, we intend to derive *conditions characterizing globally optimal solutions in the problem of maximizing the convex objective f under several convex inequality constraints*. This can be viewed as a particular case of the general situation considered by the author in ([2], [3, section III]) where a convex function was maximized over a convex set. The conditions expressed there were given in terms of the ε-subdifferential of the objective function and the ε-normal directions to the constraint set. The nice thing in the 'quadratic world' is that these global optimality conditions can be exploited in the calculations.

We do not address here the questions of semidefinite relaxation, complexity, or obtaining good approximations of $(\mathcal{P})$.

2. A global optimality condition

Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and let C be a (nonempty) closed convex set in $\mathbb{R}^n$. Two mathematical objects are useful in deriving global optimality conditions in the problem of maximizing f over C: the so-called ε-subdifferential of f and the set of ε-normal directions to C. For $\varepsilon \geqslant 0$, the *ε-subdifferential of f at $\bar{x}$*, denoted as $\partial_\varepsilon f(\bar{x})$, is the set of (slopes) $s \in \mathbb{R}^n$ satisfying

$$f(x) \geqslant f(\bar{x}) + \langle s, x - \bar{x} \rangle - \varepsilon \text{ for all } x \in \mathbb{R}^n. \tag{1}$$

The set of *ε-normal directions to C at $\bar{x} \in C$*, denoted as $N_\varepsilon(C, \bar{c})$, is the set of (directions) $d \in \mathbb{R}^n$ satisfying

$$\langle d, x - \bar{x} \rangle \leqslant \varepsilon \text{ for all } x \in C. \tag{2}$$

For properties of ε-subdifferentials of convex functions and ε-normal directions to convex sets, we refer to [5, chapter XI].

The following general result characterizes a global maximizer $\bar{x} \in C$ of f over C.

THEOREM 1. ([2])
$\bar{x} \in C$ is a global maximizer of f over C if and only if

$$\partial_\varepsilon f(\bar{x}) \subset N_\varepsilon(C, \bar{x}) \textit{ for all } \varepsilon > 0. \tag{3}$$

Inclusion (3) has to be checked for all $\varepsilon > 0$, a priori. Instead of using the rough definitions (1), (2), an alternate way of exploiting Theorem 1 is to go through the support functions of $\partial_\varepsilon f(\bar{x})$ and $N_\varepsilon(C, \bar{x})$. The support function of $\partial_\varepsilon f(\bar{x})$, denoted as $f'_\varepsilon(\bar{x}, .)$, is the so-called *$\varepsilon$-directional derivative of f at $\bar{x}$*:

$$d \in \mathbb{R}^n \mapsto f'_\varepsilon(\bar{x}, d) = \inf_{t>0} \frac{f(\bar{x} + td) - f(\bar{x}) + \varepsilon}{t}. \tag{4}$$

Also the support function of $N_\varepsilon(C, \bar{x})$, denoted as $(I_C)'_\varepsilon(\bar{x}, .)$, is the ε-directional derivative of the indicator function I_C at $\bar{x}$:

$$d \in \mathbb{R}^n \mapsto (I_C)'_\varepsilon(\bar{x}, d) = \inf \left\{ \frac{\varepsilon}{t} : t > 0, \bar{x} + td \in C \right\}. \tag{5}$$

So, instead of writing the inclusions (3) between sets, we write the following inequalities between support functions:

$$f'_\varepsilon(\bar{x}, d) \leqslant (I_C)'_\varepsilon(\bar{x}, d) \text{ for all } d \in \mathbb{R}^d \text{ and all } \varepsilon > 0. \tag{6}$$

The trick now is to exchange the quantifiers: 'for all $d \in \mathbb{R}^n$' and 'for all $\varepsilon > 0$'. The strategy is to exploit thoroughly the condition

$$f'_\varepsilon(\bar{x}, d) \leqslant (I_C)'_\varepsilon(\bar{x}, d) \text{ for all } \varepsilon > 0.$$

in given situations, and then let d vary in $\mathbb{R}^n$.

We begin by showing how this works in the presence of just one inequality constraint.

2.1. A GLOBAL OPTIMALITY CONDITION IN THE PRESENCE OF ONE INEQUALITY CONSTRAINT

Here is our optimization problem:

$$(\mathcal{P}) \quad \begin{cases} \text{maximize} & f(x) := \frac{1}{2}\langle Ax, x\rangle + \langle a, x\rangle + \alpha \\ \text{subject to} & x \in C := \{x \in \mathbb{R}^n : g(x) \leqslant 0\}, \end{cases}$$

where $g(x) := \frac{1}{2}\langle Qx, x\rangle + \langle b, x\rangle + c$.

We make the following assumptions on the data:

- $A \neq 0$ is positive semidefinite;
- Q is positive semidefinite;
- There exists x_0 such that $g(x_0) < 0$ (Slater's condition).

Under such assumptions:

- The boundary of C is $\{x : g(x) = 0\}$, while its interior is $\{x : g(x) < 0\}$;
- A maximizer of f on C, even a local one, lies on the boundary of C.

Calculation of $f'_\varepsilon(\bar{x}, d)$. Due to the particular form of f, the calculation of $f'_\varepsilon(\bar{x}, d)$ is fairly easy ([5, chapter XI]):

$$f'_\varepsilon(\bar{x}, d) = \langle A\bar{x} + a, d\rangle + \sqrt{2\varepsilon\langle Ad, d\rangle} \text{ for all } d \in \mathbb{R}^n \text{ and all } \varepsilon > 0.$$

Calculation of $(I_C)'_\varepsilon(\bar{x}, d)$ *at a boundary point* $\bar{x}$ *of* C. Let $\bar{x}$ satisfy $g(\bar{x}) = 0$. We wish to calculate $\inf\{\frac{\varepsilon}{t} : t > 0, \bar{x} + td \in C\}$, which amounts to determining

$$\sup\{t > 0 : g(\bar{x} + td) \leqslant 0\} = \sup\left\{t > 0 : \frac{1}{2}t^2\langle Qd, d\rangle + t\langle Q\bar{x} + b, d\rangle \leqslant 0\right\} =: \bar{t}_d.$$

We know that $\nabla g(\bar{x}) = Q\bar{x} + b \neq 0$; so three cases arise:

- $\langle Q\bar{x} + b, d\rangle > 0$: there is no $t > 0$ for which

  $$\frac{1}{2}t^2\langle Qd, d\rangle + t\langle Q\bar{x} + b, d\rangle \leqslant 0$$

  whence $\bar{t}_d = +\infty$.
- $\langle Q\bar{x} + b, d\rangle = 0$: the set of $t > 0$ for which

  $$\frac{1}{2}t^2\langle Qd, d\rangle + t\langle Q\bar{x} + b, d\rangle \leqslant 0$$

  is either empty or the whole half-line $(0, +\infty)$ (depending on whether $\langle Qd, d\rangle > 0$ or equals 0); whence $\bar{t}_d = +\infty$ again.
- $\langle Q\bar{x}+b, d\rangle < 0$: then $\bar{t}_d$ is $+\infty$ if $\langle Qd, d\rangle = 0$ and $-2\frac{\langle Q\bar{x}+b,d\rangle}{\langle Qd,d\rangle}$ if $\langle Qd, d\rangle > 0$; whence $\bar{t}_d = -2\frac{\langle Q\bar{x}+b,d\rangle}{\langle Qd,d\rangle}$ in the two cases.

Therefore the necessary and sufficient condition for global optimality (6) is rewritten as:

$$\begin{cases} \langle A\bar{x} + a, d\rangle + \sqrt{2\varepsilon\langle Ad, d\rangle} + \frac{\varepsilon}{2}\frac{\langle Qd,d\rangle}{\langle Q\bar{x}+b,d\rangle} \leqslant 0 \\ \text{for all } d \text{ satisfying } \langle Q\bar{x} + b, d\rangle < 0 \text{ and all } \varepsilon > 0. \end{cases} \tag{7}$$

The main question remains: how to get rid of the $\varepsilon > 0$? For a given d satisfying $\langle Q\bar{x} + b, d\rangle < 0$ letting $\alpha = \sqrt{\varepsilon}$, the inequality in (7) becomes:

$$\theta(\alpha) : \frac{1}{2}\frac{\langle Qd, d\rangle}{\langle Q\bar{x} + b, d\rangle}\alpha^2 + \sqrt{2\langle Ad, d\rangle}\alpha + \langle A\bar{x} + a, d\rangle \leqslant 0 \text{ for all } \alpha > 0. \tag{8}$$

$\theta(\alpha)$ is a polynomial function of degree 2, with $\theta'(0) = \sqrt{2\langle Ad, d\rangle} \leqslant 0$. Hence the condition (8) is equivalent to $\theta(\alpha) \leqslant 0$ for all $\alpha \in \mathbb{R}$, which in turn is checked as:

$$\Delta(d) := \langle A\bar{x} + a, d\rangle\langle Qd, d\rangle - \langle Q\bar{x} + b, d\rangle\langle Ad, d\rangle \leqslant 0. \tag{9}$$

Consequently, the condition (7) no longer contains $\varepsilon > 0$:

$$\Delta(d) \leqslant 0 \text{ for all } d \text{ satisfying } \langle Q\bar{x} + b, d\rangle < 0. \tag{10}$$

But $\Delta(-d) = -\Delta(d)$ for all d, so that condition (10) can be read again in the form below.

THEOREM 2. *Under the hypotheses posed at the beginning of the subsection, $\bar{x}$ is a global maximizer in $(\mathcal{P}_1)$ if and only if:*

$$\begin{cases} \langle Ad, d\rangle - \langle Qd, d\rangle \frac{\langle A\bar{x}+a,d\rangle}{\langle Q\bar{x}+b,d\rangle} \leqslant 0 \\ \textit{for all } d \textit{ satisfying } \langle Q\bar{x}+b, d\rangle \neq 0. \end{cases} \tag{11}$$

There is no Lagrange-KKT multiplier showing up in (11); actually the first-order necessary condition for maximality is contained in (11) in a hidden form. Recall that the tangent cone $T(C, \bar{x})$ to C at $\bar{x}$ is the half-space described as:

$$T(C, \bar{x}) = \{d \in \mathbb{R}^n : \langle Q\bar{x}+b, d\rangle \leqslant 0\},$$

and that the first-order necessary condition for maximality reads as follows:

$$\langle \nabla f(\bar{x}), d\rangle = \langle A\bar{x}+a, d\rangle \leqslant 0 \text{ for all } d \in T(C, \bar{x}). \tag{12}$$

Reformulated with the help of a multiplier, (12) is equivalent to: there exists $\bar{\mu} \geqslant 0$ such that $A\bar{x}+a = \bar{\mu}(Q\bar{x}+b)$. We now can see how our condition (11) takes the (equivalent) form of the necessary and suffficient condition for global optimality as given by Moré [6] (see also Stern and Wolkowicz [8]).

THEOREM 3. *Under the hypotheses posed at the beginning of the subsection, $\bar{x}$ is a global maximizer in $(\mathcal{P}_1)$ if and only if there exists $\bar{\mu} \geqslant 0$ satisfying:*

$$\begin{cases} A\bar{x}+a = \bar{\mu}(Q\bar{x}+b); \\ -A+\bar{\mu}Q \textit{ is positive semidefinite}. \end{cases} \tag{13}$$

REMARKS 1.

- Condition (10) can easily be extended by continuity to all directions d satisfying $\langle Q\bar{x}+b, d\rangle \leqslant 0$. Thus, while (12) is just a necessary condition for local maximality of $\bar{x}$, the following mixture of first- and second-order information on the data f and g at $\bar{x}$ provides a necessary and sufficient condition for global maximality of $\bar{x}$:

$$\begin{cases} \langle Ad, d\rangle\langle Q\bar{x}+b, d\rangle - \langle Qd, d\rangle\langle A\bar{x}+a, d\rangle \geqslant 0 \\ \text{for all } d \in T(C, \bar{x}). \end{cases} \tag{14}$$

- The global optimality condition in Theorem 3 still holds in the following more general situation ([6, p.199]) : $Q \neq 0$ and $-\infty \leqslant \inf_{\mathbb{R}^n} g(x) < 0$.

2.2. GLOBAL OPTIMALITY WITH TWO INEQUALITY CONSTRAINTS

We now admit two convex quadratic inequalities:

$$(\mathcal{P}_2) \begin{cases} \text{maximize} \quad f(x) := \frac{1}{2}\langle Ax, x\rangle + \langle a, x\rangle + \alpha \\ \text{subject to} \quad x \in C := \{x \in \mathbb{R}^n : g_1(x) \leqslant 0 \text{ and } g_2(x) \leqslant 0\}, \end{cases}$$

where $g_i(x) := \frac{1}{2}\langle Q_i x, x\rangle + \langle b_i, x\rangle + c_i$ for $i = 1, 2$.

The two convex quadratic inequality constraints problem is often referred to as the CDT problem (see [7] and references therein). Such specific problems appear in designing trust region algorithms for constrained optimization.

We make the following assumptions on the data:

- $A \neq 0$ is positive semidefinite;
- Q_1 and Q_2 are positive definite (formulas are then a little simpler to derive than when Q_1 and Q_2 are just positive semidefinite);
- There exists x_0 such that $g_i(x_0) < 0$ and $g_2(x_0) < 0$ (Slater's condition).

Under such assumptions:

- The boundary of C is $\{x \in C : g_1(x) = 0 \text{ or } g_2(x) = 0\}$;
- A maximizer of f on C necessarily on the boundary of C.

The calculation of $f'_\varepsilon(\bar{x}, d)$ is the same as in the previous subsection. As for $(I_C)'_\varepsilon$ $(\bar{x}, d)$, its calculation is more tricky, we have to distinguish two cases: when both constraints are active at $\bar{x}$, or only one constraint is active at $\bar{x}$.

2.2.1. *Case where* $g_i(\bar{x}) = g_2(\bar{x}) = 0$

To calculate $(I_C)'_\varepsilon(\bar{x}, d)$ for $d \neq 0$ we have to determine

$$\begin{aligned}
&\sup\{t > 0 : g_i(\bar{x} + td) \leqslant 0 \text{ for } i = 1, 2\} \\
&\quad = \sup\left\{t > 0 : \frac{1}{2}t^2\langle Q_i d, d\rangle + t\langle Q_i\bar{x} + b_i, d\rangle \leqslant 0 \text{ for } i = 1, 2\right\} \\
&\quad =: t_d.
\end{aligned}$$

By exploring the various cases depending on whether or not

$$\langle \nabla g_i(\bar{x}), d\rangle = \langle Q_i\bar{x} + b_i, d\rangle < 0,$$

we easily obtain the following:

$$\begin{aligned}
&t_d = +\infty \text{ if either } \langle Q_1\bar{x} + b_1, d\rangle \geqslant 0 \text{ or } \langle Q_2\bar{x} + b_2, d\rangle \geqslant 0; \\
&t_d = \min\left\{-2\frac{\langle Q_1\bar{x} + b_1, d\rangle}{\langle Q_1 d, d\rangle}, -2\frac{\langle Q_2\bar{x} + b_2, d\rangle}{\langle Q_2 d, d\rangle}\right\} \text{ if both} \\
&\qquad \langle Q_1\bar{x} + b_1, d\rangle \text{ and } \langle Q_2\bar{x} + b_2, d\rangle \text{ are } < 0.
\end{aligned}$$

Consequently the necessary and sufficient condition for global optimality (6) becomes:

$$\begin{cases}
\langle A\bar{x} + a, d\rangle + \sqrt{2\varepsilon\langle Ad, d\rangle} + \frac{\varepsilon}{2}\min\left\{\frac{\langle Q_1 d, d\rangle}{\langle Q_1\bar{x} + b_1, d\rangle}, \frac{\langle Q_2 d, d\rangle}{\langle Q_2\bar{x} + b_2, d\rangle}\right\} \leqslant 0 \\
\text{for all } d \text{ satisfying } \langle Q_1\bar{x} + b_1, d\rangle < 0 \text{ and } \langle Q_2\bar{x} + b_2, d\rangle < 0 \text{ and all } \varepsilon > 0.
\end{cases} \tag{15}$$

In the present case the tangent cone $T(C, \bar{x})$ to C at $\bar{x}$ is described as

$$T(C, \bar{x} = \{d \in \mathbb{R}^n : \langle Q_1\bar{x} + b_1, d\rangle \leqslant 0 \text{ and } \langle Q_2\bar{x} + b_2, d\rangle \leqslant 0\},$$

and the first-order necessary condition for maximality of $\bar{x}$ reads as follows:

$$\langle A\bar{x} + a, d\rangle \leqslant 0 \text{ for all } d \in T(C, \bar{x}). \tag{16}$$

We get rid of the $\varepsilon > 0$ in (15) as we have done in the previous subsection in the situation where only one inequality constraint was present. We skip details of calculation and directly give a necessary and sufficient condition for global optimality in a form parallel to (14).

THEOREM 4. *Under the assumptions posed at the beginning of the subsection, $\bar{x}$ satisfying $g_1(\bar{x}) = g_2(\bar{x}) = 0$ is a global maximizer in $(\mathcal{P}_2)$ if and only if*

$$\begin{cases} \langle Ad, d\rangle\langle Q_1\bar{x} + b_1, d\rangle\langle Q_2\bar{x} + b_2, d\rangle \\ -\langle A\bar{x} + a, d\rangle \min\{\langle Q_1 d, d\rangle\langle Q_2\bar{x} + b_2, d\rangle, \langle Q_2 d, d\rangle\langle Q_1\bar{x} + b_1, d\rangle\} \leqslant 0 \\ \textit{for all } d \in T(C, \bar{x}). \end{cases} \tag{17}$$

We know that, reformulated with the help of Lagrange -KKT multipliers $\bar{\mu}_i$, condition (16) is equivalent to the following: there exist $\bar{\mu}_1 \geqslant 0$ and $\bar{\mu}_2 \geqslant 0$ such that:

$$A\bar{x} + a = \bar{\mu}_1(Q_1\bar{x} + b_1) + \bar{\mu}_2(Q_2\bar{x} + b_2). \tag{18}$$

Now, since the interior of $T(C, \bar{x})$ consists of those $d \in \mathbb{R}^n$ for which both $\langle Q_1\bar{x} + b_1, d\rangle$ and $\langle Q_2\bar{x} + b_2, d\rangle$ are <0, a necessary and sufficient condition for global optimality written with the multipliers $\bar{\mu}_1$, $\bar{\mu}_2$, and parallel to (10) (or (11)) is as follows.

THEOREM 5. *Under the assumptions posed at the beginning of the subsection, $\bar{x}$ satisfying $g_1(\bar{x}) = g_2(\bar{x}) = 0$ is a global maximizer in $(\mathcal{P}_2)$ if and only if there exist $\bar{\mu}_1 \geqslant 0$ and $\bar{\mu}_2 \geqslant 0$ satisfying:*

$$A\bar{x} + a = \bar{\mu}_1(Q_1\bar{x} + b_1) + \bar{\mu}_2(Q_2\bar{x} + b_2); \tag{19}$$

$$\begin{aligned} &\langle Ad, d\rangle - \bar{\mu}_1 \max\left\{\langle Q_1 d, d\rangle, \frac{\langle Q_1\bar{x} + b_1, d\rangle}{\langle Q_2\bar{x} + b_2, d\rangle}\langle Q_2 d, d\rangle\right\} \\ &-\bar{\mu}_2 \max\left\{\langle Q_2 d, d\rangle, \frac{\langle Q_2\bar{x} + b_2, d\rangle}{\langle Q_1\bar{x} + b_1, d\rangle}\langle Q_1 d, d\rangle\right\} \leqslant 0 \\ &\textit{for all } d \in int T(C, \bar{x}). \end{aligned} \tag{20}$$

Condition (20) looks like a second order condition for maximality, actually mixing first and second (differential) information on the data f, g_1 and g_2 at $\bar{x}$. Let $K := int T(C, \bar{x})$ and let $H_2(\bar{x}, \bar{\mu}_1, \bar{\mu}_2; .)$ denote the homogeneous (of degre two) function occurring in (20). We indeed have:

$$H_2(\bar{x}, \bar{\mu}_1, \bar{\mu}_2, d) \leqslant 0 \text{ for all } d \in K \cup (-K), \tag{21}$$

but, contrary to the case where only one inequality constraint was present (Theorems 2 and 3), the closure of $K \cup (-K)$ is not the whole $\mathbb{R}^n$, whence (17) or (21) cannot be expanded to all directions d in $\mathbb{R}^n$.

$H_2(\bar{x}, \bar{\mu}_1, \bar{\mu}_2, .)$ can be compared to the quadratic form associated with the Hessian matrix of the (usual) Lagrangean function:

$$x \mapsto \mathcal{L}(x, \bar{\mu}_1, \bar{\mu}_2) := f(x) - \bar{\mu}_1 g_1(x) - \bar{\mu}_2 g_2(x)$$

at $\bar{x}$. Indeed

$$H_2(\bar{x}, \bar{\mu}_1, \bar{\mu}_2, d) \leqslant \langle Ad, d\rangle - \bar{\mu}_1\langle Q_1 d, d\rangle - \bar{\mu}_2\langle Q_2 d, d\rangle \tag{22}$$

$$= \langle \nabla^2_{xx}\mathcal{L}(\bar{x}, \bar{\mu}_1, \bar{\mu}_2)d, d\rangle \text{ for all } d \in \mathbb{R}^n. \tag{23}$$

One thus recovers from Theorem 5 the following well-known *sufficient* condition for global maximality: if, for $\bar{x}$ satisfying $g_1(\bar{x}) = g_2(\bar{x}) = 0$, there exist $\bar{\mu}_1 \geqslant 0$ and $\bar{\mu}_2 \geqslant 0$ such that (18) holds true and $A - \bar{\mu}_1 Q_1 - \bar{\mu}_2 Q_2$ is negative semidefinite, then $\bar{x}$ is a global maximizer of $(\mathcal{P}_2)$.

Note also that a (sharpened) *necessary* condition for global maximality was obtained in [7, p. 589]: under the assumptions of our problem, it states that if $\bar{x}$ is a global maximizer of $(\mathcal{P}_2)$, then there exist $\bar{\mu}_1 \geqslant 0$ and $\bar{\mu}_2 \geqslant 0$ such that (18) holds true and $A - \bar{\mu}_1 Q_1 - \bar{\mu}_2 Q_2$ has at most one strictly positive eigenvalue. Deriving the latter property from (20) is not straightforward.

2.2.2. *Case where* $g_1(\bar{x}) = 0$ *and* $g_2(\bar{x}) < 0$

Here again the tangent cone $T(C, \bar{x})$ to C at $\bar{x}$ is the half-space

$$T(C, \bar{x}) = \{d \in \mathbb{R}^n : \langle Q_1\bar{x} + b_1, d\rangle \leqslant 0\}.$$

Even if only the first constraint function g_1 is active at $\bar{x}$, the second one g_2 necessarily plays a role, since we are considering a *global* optimality condition at $\bar{x}$. To calculate $(I_C)'_\varepsilon(\bar{x}, d)$ for $d \neq 0$, we need to determine

$$\begin{aligned} t_d &:= \sup\{t > 0 : g_i(\bar{x} + td) \leqslant 0 \text{ for } i = 1, 2\} \\ &= \sup\left\{t > 0 : \frac{1}{2}t^2\langle Q_1 d, d\rangle + t\langle Q_1\bar{x} + b_1, d\rangle \leqslant 0 \text{ and}\right. \\ &\quad \left.\frac{1}{2}t^2\langle Q_2 d, d\rangle + t\langle Q_2\bar{x} + b_2, d\rangle + g_2(\bar{x}) \leqslant 0\right\}. \end{aligned}$$

The first inequality in the definition of t_d is treated as in Section 2.2.1. It gives rise to

$$t_d^{(1)} = \begin{cases} +\infty & \text{if } \langle Q_1\bar{x} + b_1, d\rangle \geqslant 0, \\ -2\frac{\langle Q_1\bar{x}+b_1,d\rangle}{\langle Q_1 d,d\rangle} & \text{if } \langle Q_1\bar{x} + b_1, d\rangle < 0. \end{cases} \tag{24}$$

Handling the second inequality in the definition of t_d leads us to

$$t_d^{(2)} = -2\,\frac{\mathcal{R}_2(\bar{x}, d)}{\langle Q_2 d, d\rangle},$$

where

$$2\mathcal{R}_2(\bar{x}, d) := \langle Q_2\bar{x} + b_2, d\rangle - [(\langle Q_2\bar{x} + b_2, d\rangle)^2 - 2\langle Q_2 d, d\rangle g_2(\bar{x})]^{1/2}.$$

($\mathcal{R}_2(\bar{x}, d)$ is a sort of 'residual term' due to the fact that g_2 is not active at $\bar{x}$; $\mathcal{R}_2(\bar{x}, d) < 0$ for $d \neq 0$). Hence,

$$t_d = \min(t_d^{(1)}, t_d^{(2)}) \tag{25}$$

$$= \begin{cases} +\infty & \text{if } \langle Q_1\bar{x} + b_1, d\rangle \geqslant 0, \\ \min\left\{-2\frac{\langle Q_1\bar{x}+b_1,d\rangle}{\langle Q_1 d,d\rangle}, -2\,\frac{\mathcal{R}_2(\bar{x},d)}{\langle Q_2 d,d\rangle}\right\} & \text{if } \langle Q_1\bar{x} + b_1, d\rangle < 0. \end{cases} \tag{26}$$

Consequently the necessary and sufficient condition for global optimality (6) is reformulated as:

$$\begin{cases} \langle A\bar{x} + a, d\rangle + \sqrt{2\varepsilon\langle Ad, d\rangle} + \frac{\varepsilon}{2}\min\left\{\frac{\langle Q_1 d,d\rangle}{\langle Q_1\bar{x}+b_1,d\rangle}, \frac{\langle Q_2 d,d\rangle}{\mathcal{R}_2(\bar{x},d)}\right\} \leqslant 0 \\ \text{for all } d \text{ satisfying } \langle Q_1\bar{x} + b_1, d\rangle < 0 \text{ and all } \varepsilon > 0. \end{cases} \tag{27}$$

We get rid of the $\varepsilon > 0$ in (27) as in the previous subsections. This leads us to the following analog of Theorem 4.

THEOREM 6. *Under the assumptions posed at the beginning of the subsection, $\bar{x}$ satisfying $g_1(\bar{x}) = 0$ and $g_2(\bar{x}) < 0$ is a global maximizer in $(\mathcal{P}_2)$ if and only if*

$$\begin{cases} \langle Ad, d\rangle\langle Q_1\bar{x} + b_1, d\rangle\mathcal{R}_2(\bar{x}, d) \\ -\langle A\bar{x} + a, d\rangle \min\{\langle Q_1 d, d\rangle\mathcal{R}_2(\bar{x}, d), \langle Q_2 d, d\rangle\langle Q_1\bar{x} + b_1, d\rangle\} \leqslant 0 \\ \textit{for all } d \in T(C, \bar{x}). \end{cases} \tag{28}$$

Note that the expression above is no longer symmetric in d (since $\mathcal{R}_2(\bar{x}, -d) \neq -\mathcal{R}_2(\bar{x}, d)$).

Reformulated with the help of the Lagrange -KKT multiplier $\bar{\mu}_1$ associated with the constraint function g_1 active at $\bar{x}$, we get from Theorem 6 the following analog of Theorem 5.

THEOREM 7. *Under the same assumptions as before, $\bar{x}$ satisfying $g_1(\bar{x}) = 0$ and $g_2(\bar{x}) < 0$ is a global maximizer in $(\mathcal{P}_2)$ if and only if there exists $\bar{\mu}_1 \geqslant 0$ satisfying:*

$$A\bar{x} + a = \bar{\mu}_1(Q_1\bar{x} + b_1); \tag{29}$$

$$\langle Ad, d\rangle - \bar{\mu}_1 \max\left\{\langle Q_1 d, d\rangle, \frac{\langle Q_1\bar{x} + b_1, d\rangle}{\mathcal{R}_2(\bar{x}, d)}\langle Q_2 d, d\rangle\right\} \leqslant 0$$
$$\textit{for all } d \in \mathit{int}\,T(C, \bar{x}). \tag{30}$$

2.3. A GLOBAL OPTIMALITY CONDITION IN THE PRESENCE OF m INEQUALITY CONSTRAINTS

The general problem to be considered is:

$$(\mathcal{P}_m)\quad \begin{cases} \text{maximize} & f(x) := \frac{1}{2}\langle Ax, x\rangle + \langle a, x\rangle + \alpha \\ \text{subject to} & x \in C := \{x \in \mathbb{R}^n : g_1(x) \leqslant 0, \ldots, g_m(x) \leqslant 0\}, \end{cases}$$

where $g_i(x) := \frac{1}{2}\langle Q_i x, x\rangle + \langle b_i, x\rangle + c_i$ for $i = 1, 2, \ldots, m$.
We assume the following on the data:

- $A \neq 0$ is positive semidefinite;
- $Q_1, Q_2, \ldots, Q_m$ are positive definite;
- There exists x_0 such that $g_i(x_0) < 0$ for all i (Slater's condition).

For $\bar{x} \in C$, let $I(\bar{x}) := \{i : g_i(\bar{x}) = 0\}$. We have that

$$T(C, \bar{x}) = \{d \in \mathbb{R}^n : \langle Q_i\bar{x} + b_i, d\rangle \leqslant 0 \text{ for all } i \in I(\bar{x})\},$$
$$intT(C, \bar{x}) = \{d \in \mathbb{R}^n : \langle Q_i\bar{x} + b_i, d\rangle < 0 \text{ for all } i \in I(\bar{x})\}.$$

For $i \notin I(\bar{x})$, we note $\mathcal{R}_i(\bar{x}, d)$ the associated residual term, that is defined as:

$$2\mathcal{R}_i(\bar{x}, d) := \langle Q_i\bar{x} + b_i, d\rangle - [(\langle Q_i\bar{x} + b_i, d\rangle)^2 - 2\langle Q_i d, d\rangle g_i(\bar{x})]^{1/2}.$$

The method we now follow is the one developed in full detail when $m = 2$ in Section 2.2. We skip over calculations and present the final statement.

THEOREM 8. *Under the assumptions posed in this subsection, $\bar{x} \in C$ is a global maximizer in $(\mathcal{P}_m)$ if and only if there exist $\bar{\mu}_j \geqslant 0$, $j \in I(\bar{x})$, such that:*

$$A\bar{x} + a = \sum_{j \in I(\bar{x})} \bar{\mu}_j (Q_j\bar{x} + b_j);$$

$$\langle Ad, d\rangle - \sum_{j \in I} \bar{\mu}_j(\bar{x}) \min\left\{\langle Q_j d, d\rangle, \frac{\langle Q_j\bar{x} + b_j, d\rangle}{\langle Q_i\bar{x} + b_i, d\rangle}, i \in I(\bar{x}), i \neq j;\right.$$
$$\left.\frac{\langle Q_j\bar{x} + b_j, d\rangle}{\mathcal{R}_i(\bar{x}, d)}\langle Q_i d, d\rangle, i \notin I(\bar{x}\right\} \leqslant 0$$

for all $d \in intT(C, \bar{x})$.

3. Conclusion

For our nonconvex quadratic optimization problems, global optimality conditions consist in combinations of two conditions:

- the classical first order condition for optimality;
- a complementary condition stating that some homogeneous (of degree two) function, mixing first and second order (differential) information about the data, should have a constant sign on a convex cone.

We agree that it is hard to assess the practical use of these global optimality conditions. The situation was the same some years ago for optimizing a convex quadratic function over a convex polyhedral set; several algorithms have now been designed to use these global optimality conditions. We expect the conditions presented here to provide material for future work in the same direction.

References

1. C. Floudas and V. Visweswaran, Quadratic optimization, in *Handbook for global optimization*, Kluwer Academic Publishers, Dordrecht, 1995, pp. 217–269.
2. J.-B. Hiriart-Urruty, From convex to nonconvex optimization, necessary and sufficient conditions for global optimality, in *Nonsmooth optimization and related topics*, Plenum, New York 1989, pp. 219–239.
3. J.-B. Hiriart-Urruty, Conditions for global optimality, in *Handbook for global optimization*, Kluwer Academic Publishers, Dordrecht, 1995, pp. 1–26.
4. J.-B. Hiriart-Urruty, Global optimality conditions 2, *Journal of Global Optimization* 13 (1998) 349–367.
5. J.-B. Hiriart-Urruty and C. Lemaréchal, Convex analysis and minimization algorithms, *Grundlehren der mathematischen Wissenchaften 305 & 306.* Springer, Berlin, Heidelberg, 1993.
6. J. Moré, Generalizations of the trust region problem, *Optimization Methods & Software* 2 (1993) 189–209.
7. J.-M. Peng and Y.-X. Yuan, Optimality conditions for the minimization of a quadratic with two quadratic constraints, *SIAM Joournal on Optimization* 7 (1997) 576–594.
8. R. Stern and H. Wolkowicz, Indefinite trust region subproblems and nonsymmetric eigenvalue perturbations, *SIAM Journal on Optimization* 5 (1995) 286–313.